Семинар по релятивистской астрофизике. 17 апреля 2008

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Аннотация

Немного аппарата: символы Кристоффеля из метрики, тензоры Римана, Риччи. Физика (в виде набросков): Ур. Эйнштейна и вариационный принцип Гильберта.

Опять будет смесь текстов на русском и английском.

1 Повторим Обозначения

Примем следующие соглашения.
4D индексы латинские, например, $i = 0, 1, 2, 3$.
3D индексы греческие, например, $\alpha = 1, 2, 3$.
Метрика в общем случае:

$$ds^2 = g_{ik} dx^i dx^k \ .$$

В этой записи g_{ik} означает компоненты метрического тензора. Иногда для обозначения тензоров как геометрических объектов вводят специальные жирные шрифты, тогда $\mathbf{g}$ обозначает метрический тензор независимо от координат (по аналогии с векторами). Я считаю, что удобней писать g_{ab} для координатно-независимых обозначений тензоров (используя первые буквы латинского алфавита), так как тут сразу видно, какой тензор: контравариантный, ковариантный, смешанный и т.п.
Я обозначил производную по параметру обратным штрихом:

$$\grave{x}^k \equiv \frac{dx^k}{d\lambda}$$

Формула преобразования контравариантных тензоров (фактически, определение тензора) от координат x к $\tilde{x}$:

$$\tilde{A}^{ik}(\tilde{x}) = A^{mn}(x)\frac{\partial \tilde{x}^i}{\partial x^m}\frac{\partial \tilde{x}^k}{\partial x^n} \ ,$$

ковариантных

$$\tilde{A}_{ik}(\tilde{x}) = A_{mn}(x)\frac{\partial x^m}{\partial \tilde{x}^i}\frac{\partial x^n}{\partial \tilde{x}^k} \ ,$$

и т.п.
В *3D* часто пишем

$$d\ell^2 = g_{\alpha\beta}dx^{\alpha\beta}$$

Сигнатура в четырёхмерии $(+---)$, т.е., например,

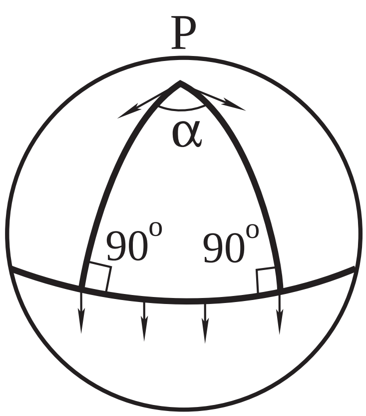
$$ds^2 = c^2dt^2 - d\ell^2 \ .$$

Скорость света c часто полагается равной единице, что видно из контекста.

2 Параллельный перенос и ковариантные производные

2.1 Избыток суммы углов треугольника – мера кривизны

Let us return to the example of a 2D sphere. We can use geodesics (i.e. great circles on it) to build triangles etc. and to find the curvature. Consider a triangle on a sphere with one vertex in the pole, and two other on the equator.



The sum of angles is not π, it is $\Sigma_\triangle = \pi+\alpha$, that is the geometry on the surface is not Euclidean. The area of this triangle is $S = \alpha R^2$. If we take another, very small, triangle on this surface with all sides much less than R and small S, we will see that it looks like almost Euclidean triangle. The excess $\Sigma_\triangle - \pi$ almost vanishes for small S, we find $\Sigma_\triangle - \pi \propto S \to 0$, when $S \to 0$, but the ratio

$$\frac{\Sigma_\triangle - \pi}{S} = \frac{1}{R^2}$$

remains finite, the same as in the big triangle on the figure. So if we were 2D creatures living on a sphere we could measure

the curvature $1/R^2$ of our world if we measure the excess $\Sigma_\triangle - \pi$ accurately. Такой избыток (или недостаток) суммы углов можно намерить, не выходя из 2D пространства, если носить вектор по контуру треугольника параллельно самому себе. При возврате в исходную точку вектор повернётся как раз на этот избыток. The same can be done in 3D space and in 4D spacetime. In this way we measure the *intrinsic* curvature of the space. (E.g. Lobachevsky tried to measure the difference $\Sigma_\triangle - \pi$. Note that in Lobachevsky's geometry the curvature is negative, so the sum of three angles $\Sigma_\triangle$ is less than π).
We can build expressions for l , for $\Sigma_\triangle$ through g_{ik} and find that zero curvature imposes some relations on g_{ik} , and on their derivatives. Now if we construct *another* 3D space with

$$ds^2 = \sum_{i,k} g_{ik} d\xi^i d\xi^k \equiv g_{ik} d\xi^i d\xi^k \ ,$$

but with $g_{ik}(\xi)$ not obeying the relations of the flat space, this *metric* will describe some curved space. This formalism was introduced by Riemann who described a space by g_{ik}, being arbitrary functions of coordinates. In Riemann space the curvature can change from point to point. I will not develop the full formalism see the books by LL (Landau & Lifshitz), Weinberg, Utiyama MTW (Misner, Thorne & Wheeler), Wald, and also Carroll l.c., but I will tell you enough for doing most basic calculations in GR.

2.2 Символы Кристоффеля

Мы показали, что условие экстремума длины $l = \int_A^B ds = \int_A^B \sqrt{L_s} d\lambda$ эквивалентно экстремалям 'действия'

$$S_s = \int_A^B L_s d\lambda \ .$$

Уравнения Эйлера-Лагранжа (Euler-Lagrange Equations, ELE) для этого S_s:

$$\frac{d}{d\lambda}\left(\frac{\partial L_s}{\partial \dot{x}^m}\right) - \frac{\partial L_s}{\partial x^m} = 0 \qquad (1)$$

описывают геодезические. Мы назвали L_s *геодезическим лагранжианом.* Как мы видели на примере метрики Шварцшильда, во многих случаях совсем не нужно 'решать' ELE – их интегралы часто очевидны из симметрий задачи. Тем не менее, эти уравнения полезны нам для вычислений коэффициентов связности, т.е. символов Кристоффеля Γ^i_{mn}. Покажу, как это делается.
Обозначим

$$\partial_m g_{ik} \equiv \frac{\partial g_{ik}}{\partial x^m} \ .$$

Из ELE (1) следует:

$$2g_{im}\ddot{x}^i + 2\dot{x}^i\dot{x}^k\partial_k g_{im} - \partial_m g_{ik}\dot{x}^i\dot{x}^k = 0\ , \tag{2}$$

где мы использовали

$$\frac{dg_{im}}{d\lambda} = \frac{dx^k}{d\lambda}\partial_k g_{im} = \dot{x}^k\partial_k g_{im}.$$

Замечая, что k и i – это немые индексы в (2), и что

$$\dot{x}^i\dot{x}^k\partial_k g_{im} = \dot{x}^k\dot{x}^i\partial_i g_{km}\ ,$$

получаем

$$2g_{im}\ddot{x}^i + (\partial_k g_{im} + \partial_i g_{km} - \partial_m g_{ik})\,\dot{x}^i\dot{x}^k = 0\ .$$

Умножим на матрицу, обратную метрике, т.е. на $g^{mn} \equiv g^{-1}_{mn}$, которая по определению удовлетворяет условию $g_{im}g^{mn} \equiv \delta^i_n$. Тогда, разделив на 2, имеем

$$\ddot{x}^n + \frac{1}{2}g^{mn}\,(\partial_k g_{im} + \partial_i g_{km} - \partial_m g_{ik})\,\dot{x}^i\dot{x}^k = 0\ .$$

Если обозначить

$$\Gamma_{mik} \equiv \frac{1}{2}\,(\partial_k g_{im} + \partial_i g_{km} - \partial_m g_{ik})$$

и

$$\Gamma^n_{ik} \equiv g^{mn}\Gamma_{mik} \equiv \frac{1}{2}g^{mn}(\partial_k g_{im} + \partial_i g_{km} - \partial_m g_{ik})\ , \tag{3}$$

– коэффициенты связности (символы Кристоффеля 1-го и 2-го рода), то окончательно получим

$$\ddot{x}^n + \Gamma^n_{ik}\dot{x}^i\dot{x}^k = 0\ . \tag{4}$$

Это и есть **уравнение геодезической**.
Скоро мы увидим, как это уравнение – закон движения пробной частицы в гравитационном поле – получается из закона всемирного тяготения ОТО, т.е. из уравнений Эйнштейна.

2.3 Ковариантная производная

We denoted

$$\Gamma^m_{ik} \equiv \frac{1}{2}g^{mn}(\partial_i g_{kn} + \partial_k g_{ni} - \partial_n g_{ik})\ ,$$

(called Christoffel coefficients or connection), this gave geodesic equation

$$\ddot{x}^i + \Gamma^i_{nm}\dot{x}^n\,\dot{x}^m\ = 0\ .$$

In Euclidean space we can choose Cartesian coordinates in which $\Gamma^i_{nm} = 0$, and the geodesic equation is just $\ddot{x}^i \equiv d^2x^i/d\lambda^2 = 0$, which is the equation for a straight line.

Many textbooks on GR start to solve geodesic equations to find orbits in curved space. As I have shown, for many problems we get the answer directly from the metric and direct integrals of ELE. I need the geodesic equation for another purpose. I can rewrite it via 4-velocity, $u^i \equiv dx^i/ds$:

$$du^i + \Gamma^i_{nm} u^n dx^m = 0 \ .$$

This replaces the condition $du^i = 0$ of inertial motion in a flat spacetime, and in a curved time it is just $Du^i = 0$, if we give the following definition for covariant differential D for any 4-vector A^i:

$$DA^i \equiv dA^i + \Gamma^i_{nm} A^n dx^m \ .$$

Why is this definition natural?
On the day when I have discussed this in Osaka in 2000, 39 years earlier, on Apr.12, 1961, a first human, Yuri Gagarin, came to outer space for 108 minutes that shook the world. He could observe small floating bodies in his cabin in a perfect rectlinear motion with constant speed, hence with $du^i = 0$. Let us take a body with the speed 1 mm/s and let it go in the cabin of the Vostok spacecraft for 10^3 seconds, so its path was only 1 m. But relative to a point on Earth (e.g. relative to Osaka) the same body made almost 10^4 km (!) on the path that looks curved, but actually is a geodesic, a straight line in the space-time. To describe this we use $Du^i = 0$. The path of the small body could be along the geodesic of the CM of the Vostok, could be perpendicular to it etc. In all cases we have $Du^i = 0$, i.e. u^i is **parallel** transported in the curved space-time. Thus our definition of D takes into account the problem of parallel transport of vectors in non-Euclidean case. And it must work for all other vectors, since Gagarin could, in principle, measure any arbitrary vector A^i relative to u^i of small bodies in his free-falling system of reference, and this D makes a covariant definition working in any system.
Ковариантную производную тензоров определяют через D и используют разные обозначения, например:

$$\frac{DA^i}{Dx^m} \equiv A^i_{;m} = \partial_m A^i + \Gamma^i_{nm} A^n = A^i_{,m} + \Gamma^i_{nm} A^n \ ,$$

и

$$\frac{DA_i}{Dx^m} \equiv A_{i;m} = \partial_m A_i - \Gamma^n_{im} A_n = A_{i,m} - \Gamma^n_{im} A_n \ .$$

Чем больше индексов у тензора, тем больше символов Кристоффеля добавляется с плюсом для верхних индексов и с минусом для нижних. См. любой стандартный учебник по ОТО.
Найдём ковариантную производную метрики:

$$\begin{aligned} g_{ik;m} &= \partial_m g_{ik} - \Gamma^j_{im} g_{jk} - \Gamma^j_{km} g_{ij} = \partial_m g_{ik} - \Gamma_{kim} - \Gamma_{ikm} \\ &= \underline{\partial_m g_{ik}} - \frac{1}{2}\left(\underline{\partial_m g_{ik}} + \utilde{\partial_i g_{mk}} - \underline{\underline{\partial_k g_{im}}}\right) - \frac{1}{2}\left(\underline{\partial_m g_{ki}} + \underline{\underline{\partial_k g_{mi}}} - \utilde{\partial_i g_{km}}\right) = 0. \end{aligned}$$

Здесь члены, подчёркнутые одинаковыми символами, взаимно уничтожаются при учёте симметрии $g_{ik} = g_{ki}$. Отсюда для ковариантного дифференциала $Dg_{ik} = 0$ и

$$DA_i = D(g_{ik}A^k) = g_{ik}DA^k + A^k Dg_{ik} = g_{ik}DA^k \ ,$$

как и должно быть для векторов (аналогично и для любых других тензоров). ЛЛ (и Вихлинин) именно из этого свойства $g_{ik;m} = 0$ выводят связь символов Кристоффеля с метрикой.

2.4 Введение тензора кривизны

Having a definition of parallel transport of vectors, we can measure the things like the excess of angles $\Sigma_\triangle - \pi$ in a triangle, just doing a loop with a vector along the triangle and measuring the angle between the directions of initial and final vectors. In the case of a 2D-sphere let us go out from the pole with a vector along meridian. On equator we go keeping it perpendicular to the path, and then return back to the pole along another meridian. The final rotation of the vector is $\alpha = \Sigma_\triangle - \pi$. For small α

$$|\Delta\vec{A}| \approx |\vec{A}|\alpha = |\vec{A}|S/R^2,$$

where S is the triangle area, $\Delta\vec{A} \perp \vec{A}$, since the length of the vector does not change.

Via components we can write $\vec{A} = (A_1,\ A_2)$ in a 2D-space with coordinates x_1, x_2 for the parallel transport over a small contour with area Δf:

$$\Delta A_1 = \kappa\ A_2\ \Delta f,$$

$$\Delta A_2 = -\kappa\ A_1\ \Delta f,$$

i.e. the matrix of transformation is $\begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix}$. Here κ is the internal curvature (equal $\kappa = 1/R^2$ for a sphere).

In 3D, 4D etc. we have a similar formula for components of the vector but now we have to add information on the orientation of the area in space, hence we add indexes: Δf^{lm}, and write

$$\Delta A_k = \frac{1}{2} R^i_{klm} A_i\, \Delta f^{lm},$$

here R^i_{klm} is the curvature tensor.

For parallel transport $DA_k = 0$, hence the change of components of A_k is

$$\delta A_k = \Gamma^i_{km} A_i dx^m$$

then

$$\Delta A_k = \oint \delta A_k = \oint \Gamma^i_{km} A_i dx^m$$

and the contour integral reduces to the surface one over Δf^{lm} via Stokes theorem, so one gets

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl} \ ,$$

см. стандартные учебники. Наш выбор индексов и знаков соответствует ЛЛ (Ландау-Лифшицу).
Due to symmetries R^i_{klm} has 20 independent components in 4D case. Thus R^i_{klm} is expressed through g_{ik}, $\partial g_{ik}/\partial x^l$, and $\partial^2 g_{ik}/\partial x^l \partial x^m$. So if the field g_{ik} is known, then the curvature is known in every point.

3 Вычисление тензоров Риччи и т.п.

По метрике можно механически вычислить коэффициенты связности Γ^i_{mn} и тензор кривизны R^i_{klm}. Для действия Гильберта нам нужен только скаляр Риччи (Ricci)

$$R = g^{ik} R_{ik} \ ,$$

где

$$R_{ik} = R^m_{imk}$$

– тензор Риччи. Возможно и другое определение $R_{ik} = R^m_{ikm}$, а поскольку тензор Римана антисимметричен по последней паре индексов ($R^j_{ikm} = -R^j_{imk}$), знаки R_{ik} и R меняются. К сожалению, подвержен произволу и выбор знака тензора Римана R^j_{ikm}. Я стараюсь следовать определениям и соглашениям о знаках ЛЛ (Ландау-Лифшица). *Будьте внимательны здесь со знаками тензоров Римана и Риччи - они могут быть совсем различны в различных учебниках! На форзаце учебника MTW (Мизнер, Торн и Уилер) дана огромная таблица соответствия знаков в разных книгах.*

4 Equations for metric

First, we consider gravity only outside massive bodies, i.e. in vacuum. The Einstein equations that govern the behaviour of the relativistic gravitational potentials, i.e. of the metric, can be derived from the least action principle. The action for metric (called the Hilbert action) should be the integral over spacetime of a Lagrange density:

$$S_H = \int \mathcal{L}_H d^4 x \ .$$

The Lagrange density $\mathcal{L}_H$ is a tensor density, which can be written as $\sqrt{-g}$ times a scalar. The factor $\sqrt{-g} = \sqrt{-\det g_{ik}}$ simply makes a physical 4-volume out of the coordinate volume element d^4x. What scalars can be made out of the metric? The only independent scalar constructed from the metric, which is not higher than second order in its derivatives, is the Ricci scalar curvature $R = g^{ik} R_{ik}$ (where $R_{ik} = R^m_{ikm}$ is the Ricci tensor). Hilbert was the first to understand that GR can be derived from the action with this simplest possible choice for a Lagrangian, and proposed

$$\mathcal{L}_H = \frac{c^3}{16\pi G}\sqrt{-g}R \ .$$

Long ago Clifford conjectured that the empty space has its own elasticity. One can say that $\mathcal{L}$ describes this elasticity, or better to say rigidity, the tendency of the space to remain flat. The constant $c^3/16\pi G$ is high and reflects the fact that the space is weakly curved because its rigidity is high. The number $c^3/16\pi G$ has a dimension. If we say it is big, then it is not clear relative to what? The dimensionless measure is $Gm^2/\hbar c$, analogous to e.m. constant $e^2/\hbar c = 1/137$. From this we get the Planck mass

$$m_{\rm Pl} = \sqrt{c\hbar/G} \simeq 10^{-5}\text{g} \simeq 10^{19}\ \text{GeV}$$

A typical length for this mass is:

$$l_{\rm Pl} = \frac{\hbar}{m_{\rm Pl}c} = \sqrt{\frac{G\hbar}{c^3}} = 10^{-33}\ \text{cm}.$$

It is clear that $l_{\rm Pl}$ is r_g for the Planck mass. Thus the space is strongly curved at the distance $l_{\rm Pl}$ from mass $m_{\rm Pl}$. For nucleons $m \sim 10^{-24}\,\text{g} \ll m_{\rm Pl}$, $l \sim 10^{-13}\,\text{cm} \gg l_{\rm Pl}$, i.e. the effects of gravity and curvature are small in the volume of particles. This illustrates the high rigidity of vacuum.

The equations of motion, here the Einstein equations for the field g_{ik}, should come from varying the action with respect to the metric. But again, as in the case of orbits, the explicit form of equations is not needed very often, since their integrals already ‘sit’ in the Lagrangian.

Let us see this, deriving the Schwarzschild metric from Hilbert action